Assignment 5

Exercise 1

Let B be a standard Brownian motion. For each $n \geq 1$, let $B^{(n)}$ be the (random) function such that $B^{(n)}_t = B_t$ for all $t \in 2^{-n}\mathbb{N}_0$ and such that $B^{(n)}$ is linear on the intervals $[i2^{-n}, (i+1)2^{-n}]$ for all $i \geq 0$ (these processes appeared in the dyadic construction of Brownian motion). Fix $\varepsilon > 0$ and let $f\colon [0,1] \longrightarrow \mathbb{R}$ be continuous with $f(0) = 0$.

1) Show that $\mathbb{P}\big[\sup_{t\in[0,1]} |B_t - B^{(n)}_t| \leq \varepsilon/3\big] \longrightarrow 1$, as $n \to \infty$.

2) Prove that for all $n \in \mathbb{N}$, $B^{(n)}$ and $B - B^{(n)}$ are $\mathbb{P}$-independent.

3) Using uniform continuity of f, establish that $\mathbb{P}\big[\sup_{t\in[0,1]} |B_t - f(t)| \leq \varepsilon\big] > 0$.

Exercise 2

Let B be a standard Brownian motion. We will now show that $\mathbb{E}^{\mathbb{P}}\big[\sup_{t\in[0,1]} |B_t|^p\big] < +\infty$ for all $p < +\infty$

1) Show that

$$\sup_{t\in[0,1]} |B_t| \leq \sum_{n=1}^{+\infty} \sup_{i\in\{0,\dots,2^n-1\}} \big|B_{(i+1)2^{-n}} - B_{i2^{-n}}\big|.$$

2) For $p \geq 1$, deduce that

$$\mathbb{E}^{\mathbb{P}}\bigg[\sup_{t\in[0,1]} |B_t|^p\bigg]^{1/p} \leq \sum_{n=1}^{+\infty} \bigg(\sum_{i=0}^{2^n-1} \mathbb{E}^{\mathbb{P}}\big[|B_{(i+1)2^{-n}} - B_{i2^{-n}}|^p\big]\bigg)^{1/p}.$$

3) Hence deduce that $\mathbb{E}^{\mathbb{P}}\big[\sup_{t\in[0,1]} |B_t|^p\big] < +\infty$ for all $p < +\infty$ sufficiently large and therefore actually for all $p \in (0, \infty)$.

Exercise 3

For a compact set $K \subset \mathbb{R}$, we define its lower Minkowski content of dimension $d > 0$ to be

$$m_d(K) = \liminf_{n\to\infty} \frac{1}{n^d} \sum_{i\in\mathbb{Z}} \mathbf{1}_{\{K\cap[i/n,(i+1)/n]\neq\emptyset\}} \in [0, \infty].$$

Let B be a standard Brownian motion and define $K := \{t \in [0,1] \colon B_t = 0\}$. The goal of this question is to show that for $d > 1/2$, $m_d(K) = 0$, $\mathbb{P}$–a.s. (which means that the lower Minkowski dimension of K is $\leq 1/2$, $\mathbb{P}$–a.s.).

1) Show that $m_d(K)$ is measurable.

2) Prove that

$$\mathbb{E}^{\mathbb{P}}[m_d(K)] \leq \liminf_{n\to\infty} \frac{1}{n^d} \sum_{i=0}^{n-1} \mathbb{P}\big[K \cap [i/n, (i+1)/n] \neq \emptyset\big] \leq \liminf_{n\to\infty} \frac{1}{n^d} \sum_{i=0}^{n-1} \mathbb{P}\bigg[\sup_{t\in[0,1/n]} \big|B_{i/n+t} - B_{i/n}\big| \geq |B_{i/n}|\bigg].$$

3) Using the scaling and the weak Markov property of Brownian motion, show that

$$\mathbb{P}\left[\sup_{t\in[0,1/n]}\left|B_{i/n+t}-B_{i/n}\right|\geq|B_{i/n}|\right]=\mathbb{P}\left[\sup_{t\in[0,1]}|B_t|\geq\sqrt{i}|N|\right],$$

where $N\sim N(0,1)$ is independent of B.

4) Using the previous exercise and 3) above, show that for all $\alpha\in(0,1/2)$ there exists $c'_\alpha>0$ such that whenever $i\in\mathbb{N}^\star$, we have

$$\mathbb{P}\left[\sup_{t\in[0,1/n]}\left|B_{i/n+t}-B_{i/n}\right|\geq|B_{i/n}|\right]\leq c'_\alpha/i^\alpha.$$

1. Deduce that $\mathbb{E}^{\mathbb{P}}[m_d(K)]=0$ and hence $m_d(K)=0$, $\mathbb{P}$–a.s. for $d>1/2$.

Exercise 4

A function $f:D\subseteq\mathbb{R}\to\mathbb{R}$ is called locally Hölder continuous of order α at $x\in D$ if there exists $\delta>0$ and $C>0$ such that $|f(x)-f(y)|\leq C|x-y|^\alpha$ for all $y\in D$ with $|x-y|\leq\delta$. A function $f:D\subseteq\mathbb{R}\to\mathbb{R}$ is called locally Hölder continuous of order α, if it is locally Hölder continuous of order α at each $x\in D$.

1) Let $Z\sim N(0,1)$. Prove that $\mathbb{P}[|Z|\leq\varepsilon]\leq\varepsilon$ for any $\varepsilon\geq0$.

2) Prove that for any $\alpha>\frac{1}{2}$, $\mathbb{P}$–almost all Brownian paths are nowhere on $[0,1]$ locally Hölder-continuous of order α.

   Hint: take any $M\in\mathbb{N}$ satisfying $M(\alpha-\frac{1}{2})>1$ and show that the set $\{W_\cdot(\omega)$ is locally α-Hölder at some $s\in[0,1]\}$ is contained in the set

$$B:=\bigcup_{C\in\mathbb{N}}\bigcup_{m\in\mathbb{N}}\bigcap_{n=m}^{+\infty}\bigcup_{k=0}^{n-M}\bigcap_{j=1}^{M}\left\{\left|W_{\frac{k+j}{n}}(\omega)-W_{\frac{k+j-1}{n}}(\omega)\right|\leq\frac{C}{n^\alpha}\right\}.$$

3) The *Kolmogorov–Čentsov theorem* states that an $\mathbb{R}$-valued process X on $[0,T]$ satisfying

$$\mathbb{E}^{\mathbb{P}}\left[|X_t-X_s|^\gamma\right]\leq C|t-s|^{1+\beta},\ (s,t)\in[0,T]^2,$$

   where γ, β, and C are positive, has a $\mathbb{P}$-modification which is locally Hölder-continuous of order α for all $\alpha<\beta/\gamma$. Use this to deduce that Brownian motion has for every $\alpha<1/2$ a version which is locally Hölder-continuous of order α.